The calculus of Apple's elastic scrolling

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1 Introduction

Those who use Apple products often find the experience agreeable. The user interface behaves naturally thanks to its patented algorithms [4]. One of these is the so-called “elastic” or “rubber band” scrolling. This effect, often taken for granted, occurs when the user scrolls beyond the edge of the page. The page keeps moving and “bounces” back before coming to a rest. This deceptively simple animation actually involves some interesting mathematics. Inspired by a computer scientist’s blog post [7], I will attempt to reverse-engineer the elastic scrolling effect using calculus methods.

2 Inertial scrolling

Observation

When a user scrolls down a long page and lets go of the screen or touchpad, the page keeps moving but slows down and eventually stops. This is sometimes called “inertial” scrolling because the page seems to have momentum. I recorded the position of a freely-moving page sixty times per second (see Appendix A for the method). The displacement is graphed in Figure 2.1.

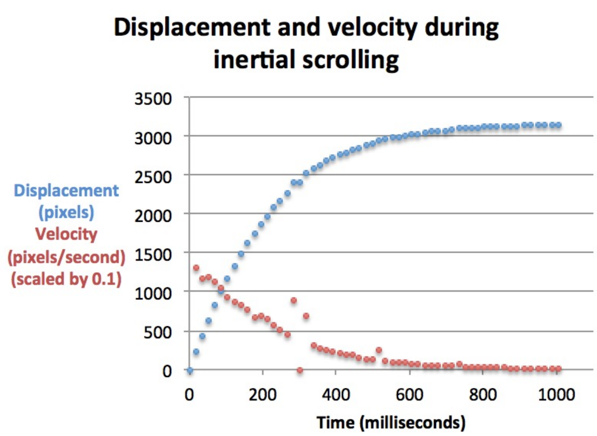

Figure 2.1

The derivative of displacement, which represents velocity, can be graphed by finding

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the slope between each adjacent pair of points.

$$v_i = \frac{x_i - x_{i-1}}{t_i - t_{i-1}}$$

Due to small glitches during recording, the displacement is slightly jagged, and this imperfection is amplified in the derivative.

Theory

How can we reproduce this behavior? Since the page slows down and eventually stops, we can infer that there is a force of friction acting on the page. Frictional force in a fluid such as air is sometimes modelled by [1]

$$F_f = -\gamma v \tag{2.1}$$

This means frictional force is proportional to velocity by the damping coefficient γ. The relationship comes from the fact that as velocity increases, there are more molecules to displace. The frictional force is opposite to the direction of motion, hence the negative (with $\gamma > 0$). Also, the unit for γ must be kg/s:

$$\begin{aligned}\gamma &= -\frac{F_f}{v} \\ &= \frac{\mathrm{kg\,m\,s^{-2}}}{\mathrm{m\,s^{-1}}} \\ &= \mathrm{kg\,s^{-1}}\end{aligned}$$

For simplicity, the mass of the page will be 1 kg throughout this essay. Thus, we can find the acceleration of the system:

$$\begin{aligned}F = ma &= -\gamma v \\ a &= -\gamma v \\ \frac{dv}{dt} &= -\gamma v\end{aligned}$$

This is a differential equation. It states that the derivative of v is a multiple of v. The only function that satisfies this is the exponential.

$$v = e^{-\gamma t}$$

A solution multiplied by a constant is also a solution (proof in Appendix B), so

$$v = v_0 e^{-\gamma t} \tag{2.2}$$

The constant v_0 is initial velocity because

$$\begin{aligned} v(0) &= v_0 e^0 \\ &= v_0 \end{aligned}$$

Initial velocity is the velocity of the page at the instant the user lets go. The velocity then decays exponentially, since $0 < e^{-\gamma} < 1$. This conclusion is consistent with Figure 2.1.

Integrating velocity gives displacement:

$$\begin{aligned} x &= \int v dt \\ &= -\frac{v_0}{\gamma} e^{-\gamma t} + c \end{aligned}$$

Initial displacement is 0, so

$$\begin{aligned} x(0) = -\frac{v_0}{\gamma} e^0 + c &= 0 \\ -\frac{v_0}{\gamma} + c &= 0 \\ c &= \frac{v_0}{\gamma} \end{aligned}$$

Therefore,

$$x = -\frac{v_0}{\gamma} e^{-\gamma t} + \frac{v_0}{\gamma} \tag{2.3}$$

Also, c is the value that x approaches as t approaches infinity:

$$\begin{aligned}&\lim_{t\to\infty} -\frac{v_0}{\gamma}e^{-\gamma t} + c\\&= -\frac{v_0}{\gamma}\cdot 0 + c\\&= c\end{aligned}$$

In practical terms, c is the total displacement of the page.

Verification

We can now confirm empirically that the theoretical displacement function is correct by manipulating it into a linear relationship.

$$\begin{aligned}x &= -\frac{v_0}{\gamma}e^{-\gamma t} + c\\x - c &= -\frac{v_0}{\gamma}e^{-\gamma t}\\e^{\gamma t} &= -\frac{v_0}{\gamma(x-c)}\\\gamma t &= \ln\left(-\frac{v_0}{\gamma(x-c)}\right)\\\gamma t &= \ln\frac{v_0}{\gamma} - \ln\left(-(x-c)\right)\\\ln\left(-(x-c)\right) &= -\gamma t + \ln\frac{v_0}{\gamma}\end{aligned} \tag{2.4}$$

So, graphing $\ln(-(x-c))$ versus t yields a linear relationship, as shown in Figure 2.2.

From Equation 2.4, the slope of the line is $-\gamma$. Therefore, the slope of the regression line shows that the damping coefficient used by Apple is approximately $5.843\,\mathrm{kg\,s^{-1}}$.

Implementation

From Equation 2.2, velocity as a function of time is

$$v(t) = v_0 e^{-\gamma t}$$

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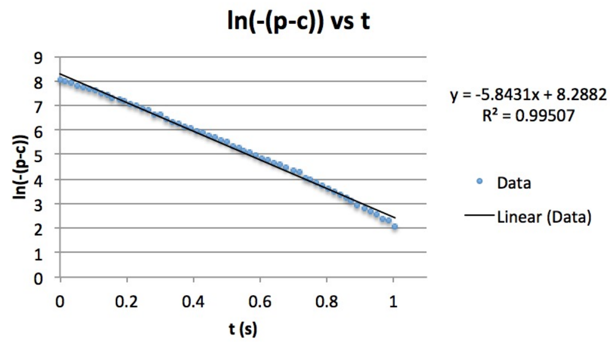

Figure 2.2

At every frame in the animation, to obtain the current velocity, the algorithm simply needs to multiply the velocity calculated in the previous frame by $e^{-\gamma n}$, where n is the time interval of each frame. This creates a geometric series where the common ratio is $e^{-\gamma n}$. As a recursive formula,

$$v_i = e^{-\gamma n} v_{i-1}$$

Then, the displacement during the i-th frame is approximated by $v_i n$. It is faster to find displacement recursively than to evaluate a closed formula at every frame.

3 Elastic scrolling

Observation

If the page is still moving when it gets to the end, it will bounce as if held by a spring. Recording the position of the page produces the graph in Figure 3.1.

Theory

A moving object attached to a spring undergoes oscillation. If the spring behaves ideally, then it exhibits simple harmonic motion modelled by sine or cosine functions. For an ideal

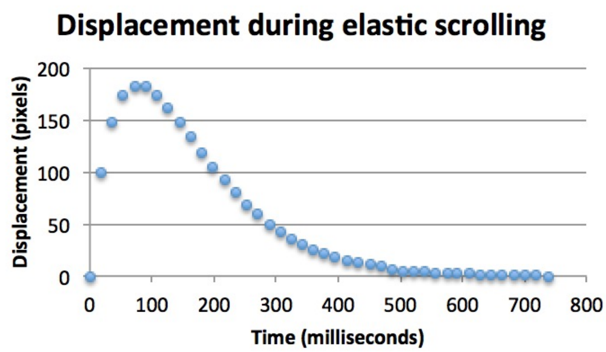

Figure 3.1

spring,

$$F_s = -kx \tag{3.1}$$

That is, the force exerted by the spring (the restoring force) is proportional to and in the opposite direction of displacement from the equilibrium, and k is the spring constant.

Without friction, simple harmonic motion can be expressed in the form

$$x = \sin(\omega t)$$

The parameter ω denotes angular velocity, in s^{-1}. It is equivalent to $\dfrac{2\pi}{T}$, where T is the period. In physics, we learn that

$$\omega = \sqrt{\frac{k}{m}}$$

The proof is outside the scope of this essay, but can be found in Appendix C. Since $m = 1\,\text{kg}$, this simplifies to

$$k = \omega^2 \tag{3.2}$$

As shown previously, the page is subject to friction, so the system undergoes *damping*. Thus, the net force is the addition of the frictional and restoring forces.

$$\begin{aligned} F &= F_f + F_s \\ ma &= -\gamma v - kx \\ a + \gamma v + \omega^2 x &= 0 \\ \frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega^2 x &= 0 \end{aligned} \tag{3.3}$$

This is a second-order linear homogeneous differential equation [2]. The derivatives add up to 0, so they must be multiples of each other. Therefore, they are exponential functions. That is,

$$x = e^{rt} \tag{3.4}$$

for some constant r. Plugging this back into 3.3,

$$\begin{aligned} \frac{d^2}{dt^2}e^{rt} + \gamma\frac{d}{dt}e^{rt} + \omega^2 e^{rt} &= 0 \\ r^2 e^{rt} + \gamma r e^{rt} + \omega^2 e^{rt} &= 0 \\ e^{rt}(r^2 + \gamma r + \omega^2) &= 0 \\ r^2 + \gamma r + \omega^2 &= 0 \end{aligned}$$

This is the characteristic equation. By the quadratic formula, the roots are

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega^2}}{2} \tag{3.5}$$

Three scenarios arise depending on the value of the discriminant:

$$\Delta = \gamma^2 - 4\omega^2 \tag{3.6}$$

If $\Delta > 0$, the system is overdamped; if $\Delta = 0$, it is critically damped; if $\Delta < 0$, it is underdamped [6]. This makes sense physically because if the damping coefficient is too

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large compared to the undamped frequency, the system will be overdamped; if the damping coefficient is too small, the system will continue to oscillate; at a perfect ratio between damping and natural frequency, the system will return to equilibrium in the shortest time. Mathematically, this makes sense because if $\Delta > 0$, there are two real roots; if $\Delta < 0$, there are two imaginary roots which lead to sinusoidal functions as the system continues to oscillate; if $\Delta = 0$, we have the (seemingly) simplest solution.

In the case of elastic scrolling, the page is critically damped since it returns to equilibrium without oscillating. Thus,

$$\begin{aligned} \gamma^2 - 4\omega^2 &= 0 \\ \gamma &= 2\omega \end{aligned} \tag{3.7}$$

We can now solve for r:

$$\begin{aligned} r &= \frac{-\gamma \pm \sqrt{0}}{2} \\ &= \frac{-2\omega}{2} \\ &= -\omega \end{aligned} \tag{3.8}$$

Using Equation 3.4, the solution to the differential equation is

$$\begin{aligned} x_1 &= e^{rt} \\ &= e^{-\omega t} \end{aligned} \tag{3.9}$$

However, this is not the only solution. In fact, in the case of repeated roots in the characteristic equation, a second solution exists in the form [3]

$$x_2 = g x_1$$

for some function $g(t)$. The motivation behind this step is essentially to find a *linearly*

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independent (Appendix B) solution. Differentiating with respect to t,

$$\begin{aligned}
x_2' &= g'x_1 + gx_1' \\
&= g'x_1 - \omega g x_1 \\
&= x_1(g' - \omega g) \\
x_2'' &= (g'x_1 + gx_1')' \\
&= g''x_1 + g'x_1' + g'x_1' + gx_1'' \\
&= g''x_1 + 2g'x_1' + gx_1'' \\
&= g''x_1 - 2\omega g'x_1 + \omega^2 g x_1 \\
&= x_1(g'' - 2\omega g' + \omega^2 g)
\end{aligned}$$

Substituting x_2 and its derivatives into the differential equation in 3.3,

$$\begin{aligned}
x_1(g'' - 2\omega g' + \omega^2 g) + \gamma x_1(g' - \omega g) + \omega^2 g x_1 &= 0 \\
x_1(g'' - 2\omega g' + \omega^2 g + \gamma(g' - \omega g) + \omega^2 g) &= 0 \\
g'' - g'(2\omega - \gamma) + g(\omega^2 - \gamma\omega + w^2) &= 0
\end{aligned}$$

From Equation 3.7, $\gamma = 2\omega$, so

$$\begin{aligned}
g'' - g' \cdot 0 + g(\omega^2 - 2\omega^2 + w^2) &= 0 \\
g'' - g' \cdot 0 + g \cdot 0 &= 0 \\
g'' &= 0
\end{aligned}$$

Integrating twice,

$$\begin{aligned}
g' &= c_1 \\
g &= c_1 t + c_2
\end{aligned}$$

Therefore, $g(t)$ is a linear function. Subsequently, the second solution can be expressed as

$$\begin{aligned} x_2 &= gx_1 \\ &= (c_1 t + c_2)e^{-\omega t} \\ &= c_1 te^{-\omega t} + c_2 e^{-\omega t} \end{aligned} \tag{3.10}$$

Sums of solutions are also solutions (Appendix B), so the general solution is [5]

$$\begin{aligned} x &= x_1 + x_2 \\ &= c_0 e^{-\omega t} + c_1 te^{-\omega t} + c_2 e^{-\omega t} \\ &= (c_0 + c_2)e^{-\omega t} + c_1 te^{-\omega t} \\ &= Ae^{-\omega t} + Bte^{-\omega t} \end{aligned} \tag{3.11}$$

for some constants A and B, which are initial conditions. These are very likely to be initial displacement and initial velocity since they affect the displacement curve. We can test this:

$$\begin{aligned} x(0) &= Ae^0 + B \cdot 0 \cdot e^0 \\ &= A \end{aligned}$$

$$\begin{aligned} v(t) &= \frac{dx}{dt} = -A\omega e^{-\omega t} + Be^{-\omega t} - B\omega te^{-\omega t} \\ v(0) &= -A\omega e^0 + Be^0 - \omega \cdot 0 \cdot e^0 \\ &= B - A\omega \end{aligned}$$

For elastic scrolling, initial displacement is 0, so

$$\begin{aligned} x_0 &= 0 = A \\ v_0 &= B - 0 \cdot \omega = B \end{aligned}$$

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Thus, displacement during elastic scrolling follows the function

$$\begin{aligned} x &= 0 \cdot e^{-\omega t} + v_0 t e^{-\omega t} \\ &= v_0 t e^{-\omega t} \end{aligned} \tag{3.12}$$

This is graphed in Figure 3.2 and is consistent with the data collected in Figure 3.1.

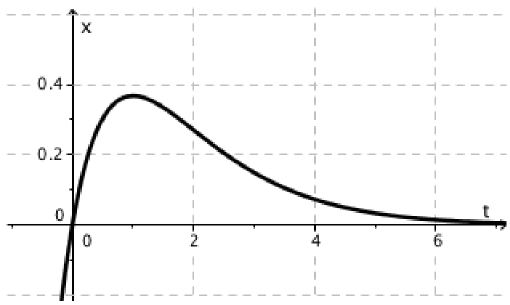

Figure 3.2: Graph of $x(t) = te^{-t}$

Verification

Because t cannot be isolated, I graphed x versus $te^{-\omega t}$, using Excel's Solver to find the ω that maximizes the R^2 value of the regression line. As shown in Figure 3.3, this yields the value of ω used by Apple, and it is approximately $12.608\,\mathrm{s}^{-1}$.

Surprisingly, this result does not satisfy $\gamma = 2\omega$ (Equation 3.7) with the previously calculated $\gamma = 5.843\,\mathrm{kg\,s}^{-1}$. The only explanation is that Apple uses different damping coefficients for inertial and elastic scrolling.

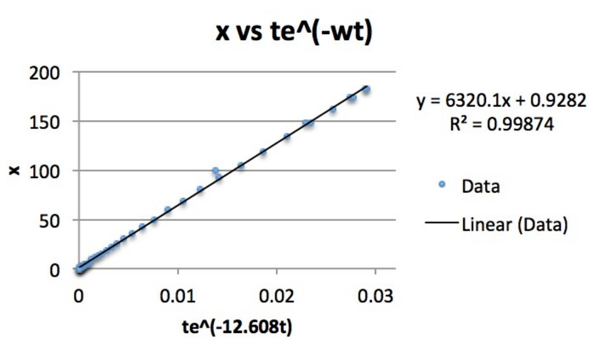

Figure 3.3

Implementation

Adapting from Equation 3.12, the closed formula for total displacement at the i-th frame is given by

$$x(i) = v_0 n i e^{-\omega n i}$$

where n is the time interval of each frame (the time elapsed is $t = ni$). We can rewrite the formula as

$$x(i) = ni \cdot g(i)$$

where g is a function defined as

$$g(i) = v_0 e^{-\omega n i}$$

Subsequently,

$$\begin{aligned} g(i+1) &= v_0 e^{-\omega n (i+1)} \\ &= v_0 e^{-\omega n i} \cdot e^{-\omega n} \\ &= g(i) \cdot e^{-\omega n} \end{aligned}$$

To find the displacement in the next frame based on values in the current frame (in other words, recursively):

$$\begin{aligned} x(i+1) &= n(i+1) \cdot g(i+1) \\ &= (ni + n) \cdot g(i) \cdot e^{-\omega n} \\ &= (ni \cdot g(i) + n \cdot g(i)) \cdot e^{-\omega n} \\ &= (x(i) + n \cdot g(i)) \cdot e^{-\omega n} \end{aligned}$$

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4 Conclusion

I have shown that Apple incorporates natural, physical phenomena into its software interfaces. These phenomena and their virtual imitations are often overlooked due to their prevalence in the world around us. Yet, artificially creating such visual effects involves fascinating mathematics. I will let the reader decide whether it is right or wrong for a company to patent the laws of physics.

References

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[7] Wittens, S. (2013, September 13). *Animate Your Way to Glory*. Retrieved December 4, 2014, from `http://acko.net/blog/animate-your-way-to-glory/`

Appendix A Recording method

I wrote the following JavaScript to record the position of a webpage in a browser while I scrolled around.

```
var X = [],
    Y = [], // arrays for coordinates
    interval,
    t0 = Date.now(); // initial time

interval = setInterval(function () { // main loop
    var t = Date.now() - t0, // time elapsed
        y = window.scrollY; // get scroll position
    X.push(t);
    Y.push(y); // insert into arrays
}, 17); // repeat at 60Hz
```

Appendix B Linear combination of solutions to differential equations

Differential equations are covered in the Math HL course, so this proof is only included as an appendix.

Given a second-order linear homogeneous differential equation in the form

$$Ay'' + By' + Cy = 0$$

where A, B and C are constants. Let the function g be a solution to this equation. That is,

$$Ag'' + Bg' + Cg = 0$$

Assuming cg is a solution for some constant c,

$$\begin{aligned}\text{LHS} &= A(cg)'' + B(cg)' + Ccg \\ &= Acg'' + Bcg' + Ccg \\ &= c(Ag'' + Bg' + Cg) \\ &= c \cdot 0 \\ &= 0 = \text{RHS}\end{aligned}$$

Therefore, a solution multiplied by a constant is also a solution to the equation.

Now, let the function h be another solution:

$$Ah'' + Bh' + Ch = 0$$

Assuming $g + h$ is a solution,

$$\begin{aligned}\text{LHS} &= A(g+h)'' + B(g+h)' + C(g+h) \\ &= Ag'' + Ah'' + Bg' + Bh' + Cg + Ch \\ &= (Ag'' + Bg' + Cg) + (Ah'' + Bh' + Ch) \\ &= 0 + 0 \\ &= 0 = \text{RHS}\end{aligned}$$

Therefore, the sum of two solutions is also a solution to the equation.

If multiples of solutions are solutions, and sums of solutions are solutions, then the sum of multiples of solutions are also solutions. In other words, linear combinations of solutions are solutions, in the form

$$y = Ag + Bh$$

Appendix C Natural frequency

If one is already aquainted with damped harmonic motion, it is straightforward to show that the undamped frequency is $\omega = \sqrt{\frac{k}{m}}$. For an ideal spring,

$$\begin{aligned} F &= -kx \\ ma + kx &= 0 \\ m\frac{d^2x}{dt^2} + kx &= 0 \end{aligned}$$

Assuming a trial solution in the form

$$x = e^{rt}$$

Substituting the trial solution into the differential equation,

$$\begin{aligned} m\frac{d^2}{dt^2}e^{rt} + ke^{rt} &= 0 \\ mr^2e^{rt} + ke^{rt} &= 0 \\ e^{rt}(mr^2 + k) &= 0 \\ mr^2 + k &= 0 \\ r &= \sqrt{-\frac{k}{m}} \\ &= i\sqrt{\frac{k}{m}} \end{aligned}$$

Plugging r into the trial solution,

$$x = e^{i\sqrt{\frac{k}{m}}t}$$

By Euler's formula,

$$\mathbb{R}(x) = \cos\left(\sqrt{\frac{k}{m}}t\right)$$

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Therefore,

$$\omega = \sqrt{\frac{k}{m}}$$